

Efficiencies

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Nomenclature

Symbol	Denotation	Unit
c	Flow speed	m/s
c_p	Specific heat at constant pressure	J/kgK
c_v	Specific heat at constant volume	J/kgK
h	Enthalpy	J/kg
n	Polytropic exponent	-
p	Pressure	Pa
R_H	Reheat factor	-
s	Entropy	J/kgK
T	Temperature	K
v	Specific volume	m ³ /kg
γ	Ratio of specific heats (also referred to as "isentropic exponent")	-
η	Efficiency	-

Subscripts

0	Total
1	Start change of state
2	End change of state
p	Polytropic
s	Isentropic
tt	Total-to-total
ts	Total-to-static

Efficiencies

In the turbomachinery context a large number of efficiencies are defined such as thermodynamic or mechanical efficiency. In the sections below the focus is put on the thermodynamic efficiencies.

For a given change of state of a fluid the efficiency is defined as the ratio between actual change in energy to ideal change in energy in case of expansion or the inverse in case of compression

- Expansion

$$\eta = \frac{\text{actual change in energy}}{\text{ideal change in energy}} \quad \text{Eq. 1}$$

- Compression

$$\eta = \frac{\text{ideal change in energy}}{\text{actual change in energy}} \quad \text{Eq. 2}$$

The symbol for efficiencies is the Greek letter η (say “eta”). For adiabatic processes the efficiency lies between 0 and 1.

Isentropic Efficiency

Depending on which process is taken as ideal process efficiencies are referred to as isentropic or polytropic efficiencies. In case of an isentropic efficiency the ideal process is represented by an isentropic change of state from start to end pressure, i.e. the same pressures as for the real process. This is illustrated in figure 1 for an expansion process by means of an enthalpy-entropy diagram (h-s diagram).

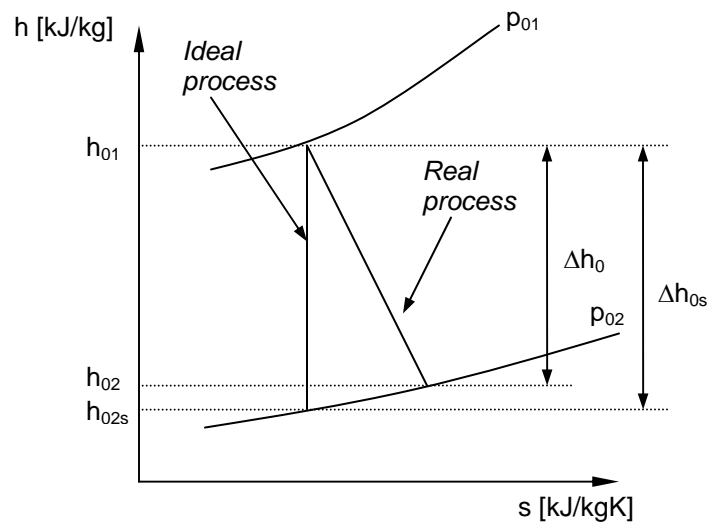


Figure 1. Expansion process

In the above depicted process the changes in total energy are referred to, which is expressed by indexing the efficiency by “tt”, i.e. “total-to-total”. Recall that the total energy is defined as follows:

$$h_0 = h + \frac{c^2}{2} \quad \text{Eq. 3}$$

The total-to-total isentropic efficiency (expansion) is thus given by

$$\eta_{tt} = \frac{\text{actual change in energy}}{\text{ideal change in energy}} = \frac{\Delta h_0}{\Delta h_{0s}} = \frac{h_{01} - h_{02}}{h_{01} - h_{02s}} \quad \text{Eq. 4}$$

In case of a compression process the situation is as follows:

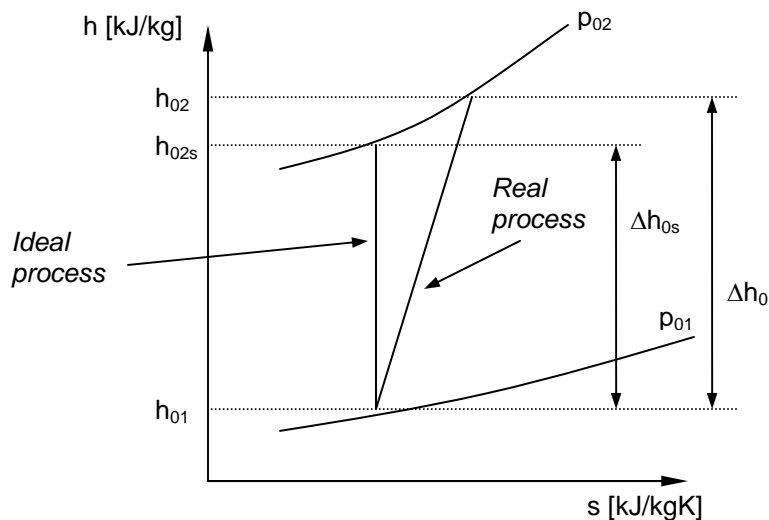


Figure 2. Compression process

Total-to-total isentropic efficiency (compression)

$$\eta_{tt} = \frac{\text{ideal change in energy}}{\text{actual change in energy}} = \frac{\Delta h_{0s}}{\Delta h_0} = \frac{h_{02s} - h_{01}}{h_{02} - h_{01}} \quad \text{Eq. 5}$$

Note:

- For adiabatic real processes the entropy must always increase during the change of state
- Due to this increase in entropy the real change in energy is smaller than the ideal during expansion. In other words, you get out less energy from the real process than you could have from an ideal one
- For the compression process the increase in entropy signifies that you need to put in more energy to compress a fluid than you would have in an ideal process

- Therefore the efficiency is always smaller or equal to unity
- The only way to reduce entropy would be to cool a process. However in such case we do no longer look into adiabatic processes

In certain cases the kinetic energy that is contained in the fluid (i.e. the amount of energy that is due to the motion) can not be used at the end of a process. An example for such a process is the last stage of a energy producing turbine where the kinetic energy in the exhaust gases is not contributing to the total energy produced. In such case a so-called total-to-static isentropic efficiency is used, identified by indexing the efficiency by "ts", i.e. "total-to-static".

An expansion line is drawn in figure 3. Note that it is necessary to include total and static states in this case.

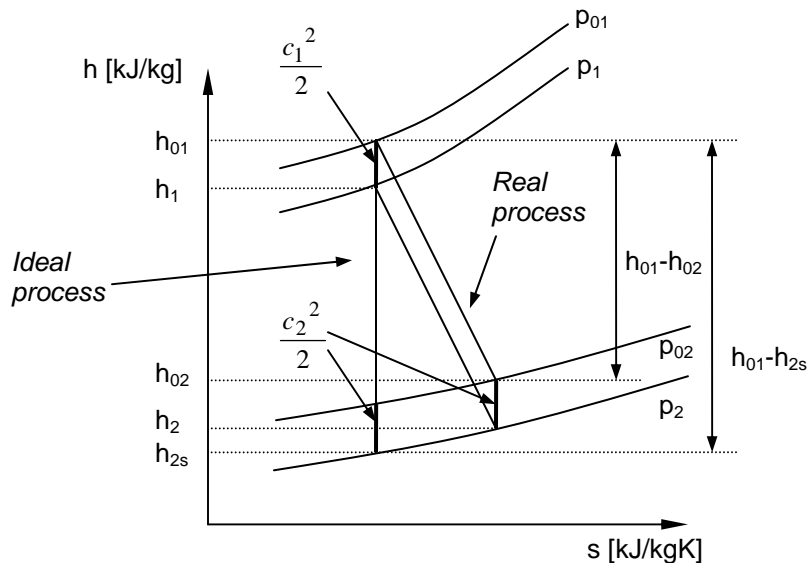


Figure 3. Expansion process; total-to-static efficiency

The total-to-total isentropic efficiency (expansion) is thus given by

$$\eta_{ts} = \frac{\text{actual change in energy}}{\text{ideal change in energy}} = \frac{h_{01} - h_{02}}{h_{01} - h_{2s}} = \frac{\Delta h_0}{\Delta h_{0s} + \frac{c_2^2}{2}} \quad \text{Eq. 6}$$

By reformulating the above expression a relation between total-to-total and total-to-static efficiency can be obtained as follows

$$\eta_{ts} = \left[\frac{\Delta h_{0s} + \frac{c_2^2}{2}}{\Delta h_0} \right]^{-1} = \left[\frac{\Delta h_{0s}}{\Delta h_0} + \frac{c_2^2}{2\Delta h_0} \right]^{-1} = \left[\frac{1}{\eta_{tt}} + \frac{c_2^2}{2\Delta h_0} \right]^{-1} \quad \text{Eq. 7}$$

This relation shows that for values of $c_2 > 0$ the total-to-static efficiency is always smaller than the total-to-total efficiency.

Calculating with Isentropic Efficiencies

Next the focus is drawn towards the calculation of efficiencies and states. For perfect gases with constant specific heat c_p the enthalpies are only a function of temperature as follows

$$h = c_p \cdot T \quad \text{Eq. 8}$$

Furthermore the gas law for a perfect gas relates temperatures and pressures for an isentropic process as given below

$$\frac{1-\gamma}{p^\gamma} T = \text{const.} \quad \text{Eq. 9}$$

, where γ is non-dimensional and stands for the ratio of specific heats

$$\gamma = \frac{c_p}{c_v} \quad \text{Eq. 10}$$

The two states 1 and 2s at the same entropy are thus related by

$$\frac{T_{01}}{T_{02s}} = \left(\frac{p_{01}}{p_{02s}} \right)^{\frac{\gamma-1}{\gamma}} \quad \text{Eq. 11}$$

By expressing T_{02s} by

$$T_{02s} = T_{01} \left(\frac{p_{02s}}{p_{01}} \right)^{\frac{\gamma-1}{\gamma}} \quad \text{Eq. 12}$$

the isentropic enthalpy difference $\Delta h_{01 \rightarrow 02s}$ can be written as

$$\Delta h_{01 \rightarrow 02s} = cp \Delta T_{01 \rightarrow 02s} = cp(T_{01} - T_{02s}) = cpT_{01} \left[1 - \left(\frac{p_{02s}}{p_{01}} \right)^{\frac{\gamma-1}{\gamma}} \right] \quad \text{Eq. 13}$$

To obtain the real change in enthalpy the efficiency must be accounted for as shown above yielding

$$\Delta h_{01 \rightarrow 02} = \eta_{it} cp T_{01} \left[1 - \left(\frac{p_{02s}}{p_{01}} \right)^{\frac{\gamma-1}{\gamma}} \right] \quad \text{Eq. 14}$$

Note that the above equation represents a rather common problem; very often the inlet state to a gas turbine is given by (p, T) , e.g. from conditions after a combustion chamber. Furthermore the exit pressure of the turbine might be set. As approximation it can also be assumed that $p_{02s} \approx p_{02}$. By knowing (or assuming) the efficiency the real change in enthalpy can thus easily be calculated.

Polytropic Efficiency

As for the isentropic efficiency the polytropic efficiency relates a real process to an ideal one. The main difference however is that the ideal process in this case is not taken as the single isentropic change of state but rather the flow work, which is defined as follows

$$y = \int v dp \quad \text{Eq. 15}$$

Thus the definition of polytropic efficiency is given by

$$\eta_p = \frac{\text{actual change in energy}}{\text{ideal change in energy}} = \frac{\Delta h_0}{y} \quad \text{Eq. 16}$$

The flow work is not easily visualized in the h-s diagram. It can be understood as infinite number of infinitesimal small isentropic changes of state that follow the real expansion line like a saw tooth curve, see figure 4. This consideration also leads to the polytropic efficiency sometimes being referred to as "small-stage efficiency".

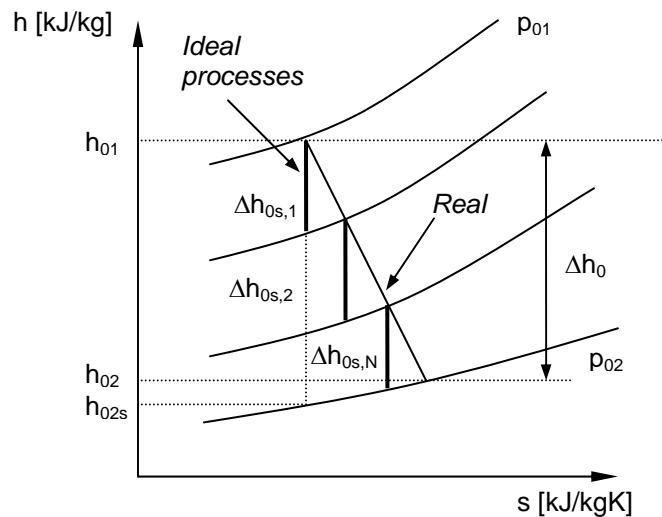


Figure 4. Illustration of flow work

Note that the sum of all these infinitesimal isentropic changes is greater than the single isentropic change from 1 to 2s.

$$\sum \Delta h_{0s,i} > \Delta h_{0s} \quad \text{Eq. 17}$$

This is due to the fact that the isobars are spread apart with increasing entropy, which in turn is due to the slope of the isobars being proportional to the temperature as follows

$$\left(\frac{\partial h}{\partial s} \right)_{p=\text{const.}} = T \quad \text{Eq. 18}$$

The growing spreading of the isobars is an indication for increased energy content of the fluid at the same pressure due to increased entropy (hint: see the entropy as measure for disorder; higher temperature leads to greater disorder). By approximating the flow work by the aforementioned infinitesimal isentropic changes a phenomenon known as “reheat” gets apparent; due to the reheating phenomenon part of the heat generated due to losses (i.e. efficiency smaller than unity) is fed back to the fluid as energy and can be used during the process. Following this consideration a reheat factor is defined as follows

$$R_H = \frac{\sum \Delta h_{0s,i}}{\Delta h_{0s}} > 1 \quad \text{Eq. 19}$$

For an expansion the isentropic and polytropic efficiencies can now be related by this reheat factor following

$$\eta_p = \frac{\Delta h_0}{\sum \Delta h_{0s,i}} = \frac{\Delta h_0}{R_H \Delta h_{0s}} = \frac{1}{R_H} \eta_{tt} \quad \text{Eq. 20}$$

As the reheat factor is larger than unity the polytropic efficiency is smaller than the isentropic efficiency.

In case of a compression the polytropic efficiency yields from

$$\eta_p = \frac{\sum \Delta h_{0s,i}}{\Delta h_0} = \frac{R_H \Delta h_{0s}}{\Delta h_0} = R_H \eta_{tt} \quad \text{Eq. 21}$$

, which leads to the polytropic efficiency being greater than the isentropic efficiency. From that point of view it is apparent that the polytropic efficiency reflects a different aspect of a change of state of a perfect gas as it takes into account the effect of reheating.

By knowing the polytropic efficiency it is possible to apply the gas law as introduced further above to polytropic changes by reformulating

$$p^{\frac{1-n}{n}} T = \text{const.} \quad \text{Eq. 22}$$

The coefficient “n” is thereby referred to as polytropic coefficient and is related to the isentropic exponent γ as follows

- Expansion process

$$\frac{n-1}{n} = \eta_p \frac{\gamma-1}{\gamma} \quad \text{Eq. 23}$$

- Compression process

$$\frac{n-1}{n} = \frac{1}{\eta_p} \frac{\gamma-1}{\gamma} \quad \text{Eq. 24}$$

Two states 1 and 2, which do not need to be at the same entropy, are now in case of an expansion process related by

$$\frac{T_1}{T_2} = \left(\frac{p_1}{p_2} \right)^{\eta_p \frac{\gamma-1}{\gamma}} \quad \text{Eq. 25}$$

Going back to the expansion sketched in figure 1 a relation between the isentropic and the polytropic efficiency and thus the reheat factor can be derived. The total-to-total isentropic efficiency was given by

$$\eta_{tt} = \frac{\Delta h_0}{\Delta h_{0s}} = \frac{\Delta h_{01 \rightarrow 02}}{\Delta h_{01 \rightarrow 02s}} \quad \text{Eq. 26}$$

It has also been shown that the isentropic change in enthalpy could directly be determined from the gas law for a perfect gas by

$$\Delta h_{01 \rightarrow 02s} = cp(T_{01} - T_{02s}) = cpT_{01} \left[1 - \left(\frac{p_{02s}}{p_{01}} \right)^{\frac{\gamma-1}{\gamma}} \right] \quad \text{Eq. 27}$$

By applying the polytropic relation the actual change in enthalpy can be obtained directly from

$$\Delta h_{01 \rightarrow 02} = cp(T_{01} - T_{02}) = cpT_{01} \left[1 - \left(\frac{p_{02}}{p_{01}} \right)^{\eta_p \frac{\gamma-1}{\gamma}} \right] \quad \text{Eq. 28}$$

Note the presence of the polytropic efficiency in the exponent reflecting the polytropic coefficient. By substituting these expressions into Eq. 26 and under assumption of $p_{02s} \approx p_{02}$ we obtain

$$\eta_{tt} = \frac{\left[1 - \left(\frac{p_{02}}{p_{01}} \right)^{\eta_p \frac{\gamma-1}{\gamma}} \right]}{\left[1 - \left(\frac{p_{02}}{p_{01}} \right)^{\frac{\gamma-1}{\gamma}} \right]} \quad \text{Eq. 29}$$

For small pressure ratios $p_{01}/p_{02} \approx 1$ the polytropic and isentropic efficiencies therefore differ very little. With increasing pressure ratio this difference also increases.

The reheat factor in case of an expansion process is then obtained from Eq. 20 as follows

$$R_H = \frac{\eta_{tt}}{\eta_p} = \frac{1}{\eta_p} \frac{\left[1 - \left(\frac{p_{02}}{p_{01}} \right)^{\eta_p \frac{\gamma-1}{\gamma}} \right]}{\left[1 - \left(\frac{p_{02}}{p_{01}} \right)^{\frac{\gamma-1}{\gamma}} \right]} \quad \text{Eq. 30}$$